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# Minimal zero sum sequences of length four over finite cyclic groups <sup>☆</sup>

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## ABSTRACT

**Text.** Let  $G$  be a finite cyclic group. Every sequence  $S$  over  $G$  can be written in the form  $S = (n_1 g) \cdots (n_l g)$  where  $g \in G$  and  $n_1, \dots, n_l \in [1, \text{ord}(g)]$ , and the index  $\text{ind}(S)$  of  $S$  is defined to be the minimum of  $(n_1 + \cdots + n_l)/\text{ord}(g)$  over all possible  $g \in G$  such that  $\langle g \rangle = \langle \text{supp}(S) \rangle$ . The problem regarding the index of sequences has been studied in a series of papers, and a main focus is to determine sequences of index 1. In the present paper, we show that if  $G$  is a cyclic of prime power order such that  $\gcd(|G|, 6) = 1$ , then every minimal zero-sum sequence of length 4 has index 1.

**Video.** For a video summary of this paper, please click [here](http://www.youtube.com/watch?v=BC7josX_xVs) or visit [http://www.youtube.com/watch?v=BC7josX\\_xVs](http://www.youtube.com/watch?v=BC7josX_xVs).

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## 1. Introduction

Let  $G$  be a finite abelian group written additively and  $\mathcal{F}(G)$  denote the free abelian monoid with basis  $G$ . The elements of  $\mathcal{F}(G)$  are called *sequences* over  $G$ . A sequence of not necessarily distinct elements from  $G$  can be written in the form  $S = g_1 \cdots g_k$ . We denote by  $\sigma(S)$  the sum of  $S$  (i.e.  $\sigma(S) = \sum_{i=1}^k g_i$ ).  $S$  is called a *zero-sum sequence* if  $\sigma(S) = 0$ . If  $S$  is a zero-sum sequence, but no proper nontrivial subsequence of  $S$  has sum zero, then  $S$  is called a *minimal zero-sum sequence*. We next introduce the concept of the index of a sequence.

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**Definition 1.1.** 1. Let  $g \in G$  be a nonzero element with  $\text{ord}(g) = n < \infty$ . For a sequence

$$S = (n_1 g) \cdot \dots \cdot (n_l g), \quad \text{where } l \in \mathbb{N}_0 \text{ and } n_1, \dots, n_l \in [1, n],$$

we define

$$\|S\|_g = \frac{n_1 + \dots + n_l}{n}.$$

Clearly,  $S$  has sum zero if and only if  $\|S\|_g \in \mathbb{N}_0$ .

2. Let  $S$  be a sequence for which  $\langle \text{supp}(S) \rangle \subset G$  is cyclic. The index of  $S$ , denoted by  $\text{ind}(S)$ , is defined by

$$\text{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } \langle \text{supp}(S) \rangle = \langle g \rangle\}.$$

3. If  $G$  is cyclic, then we let  $\text{l}(G)$  denote the smallest integer  $l \in \mathbb{N}$  such that every minimal zero-sum sequence  $S \in \mathcal{F}(G)$  of length  $|S| \geq l$  satisfies  $\text{ind}(S) = 1$ .

Throughout the paper  $G$  is always assumed to be a finite cyclic group of order  $n$ . The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. It was first addressed by Lemke-Kleitman (in the conjecture [8, page 344]), used as a key tool by Geroldinger [5, page 736], and then investigated by Gao [3] in a systematical way. Since then it has attracted a lot of attention in recent years (see for example [1,2,4,6,7,9–12]).

We first point out that the following simple technical characterizations of the index were given in [6, Lemma 5.1.2]:

$$\begin{aligned} \text{ind}(S) &= \min\{\|S\|_g \mid g \in G \text{ with } \langle \text{supp}(S) \rangle = \langle g \rangle\} \\ &= \min\{\|S\|_g \mid g \in G \text{ with } G = \langle g \rangle\}. \end{aligned}$$

It was proved independently by Savchen and Chen, and by the third author (see [10, Proposition 10], [12, Theorem 3.1] and [6, Corollary 5.1.9]) that for a finite cyclic group  $G$  of order  $n$ , if  $n \in \{1, 2, 3, 4, 7\}$ , then  $\text{l}(G) = 1$ ; otherwise,  $\text{l}(G) = \lfloor \frac{n}{2} \rfloor + 2$ . It was proved in [9] that every minimal zero-sum sequence  $S$  over  $G$  of length 1, 2, 3 has index  $\text{ind}(S) = 1$ , and if  $\gcd(|G|, 6) \neq 1$  then there exists a minimal zero-sum sequence  $S$  of length 4 such that  $S$  has  $\text{ind}(S) \neq 1$ . It was conjectured that if  $\gcd(n, 6) = 1$  then every minimal zero-sum sequence  $S$  of length 4 over  $G$  has  $\text{ind}(S) = 1$ . It can be proved by using the main result of [12] (or a result of Savchen and Chen in [10]) that there exists a minimal zero-sum sequence  $S$  of length  $l$  with  $4 < l < \lfloor \frac{n}{2} \rfloor + 2$  in a cyclic group  $G$  of order  $n$  such that  $\text{ind}(S) \neq 1$ . The only unsolved case is that whether or not every minimal zero-sum sequence of length 4 in a cyclic group  $G$  with  $\gcd(|G|, 6) = 1$  has index 1. In this paper we prove that if  $G$  is a cyclic group of prime power order such that  $\gcd(|G|, 6) = 1$ , then every minimal zero-sum sequence of length 4 has index 1. We remark that the general case when  $|G|$  is not necessarily a prime power is still open. Our main result is as follows.

**Theorem 1.2.** *Let  $G$  be a cyclic group of prime power order such that  $\gcd(|G|, 6) = 1$ , then every minimal zero-sum sequence  $S$  over  $G$  of length  $|S| = 4$  has  $\text{ind}(S) = 1$ .*

The paper is organized as follows. In the next section, we do a reduction by reducing the proof of Theorem 1.2 to that of Proposition 2.1. In Section 3 we handle the case when  $|G|$  is a prime. We then prove four important special cases of Proposition 2.1 in Section 4. The complete proof of Proposition 2.1 is given in Section 5. A related problem regarding Dedekind sums is dealt with in the last section.

## 2. Reduction

In this section we reduce the proof of Theorem 1.2 to that of Proposition 2.1.

Let  $G$  be a finite cyclic group of a prime power order  $q$ ,  $S = (x_1g)(x_2g)(x_3g)(x_4g)$ ,  $x_i \in \mathbb{N}_0$ ,  $i = 1, 2, 3, 4$ , a minimal zero-sum sequence over  $G$ . By [6, Lemma 5.1.2], we may assume that  $\text{ord}(g) = q$ ,  $1 \leq x_i \leq q-1$  and at least one of  $x_1, x_2, x_3, x_4$  is coprime with  $q$  by induction. Without loss of generality we may assume that  $\gcd(x_1, q) = 1$ . In addition, we may assume that  $x_1 \notin \{x_2, x_3, x_4\}$ . To see this, note that if we have some other integer equal to  $x_1$ , say  $S = (x_1g)(x_1g)(x_2g)(x_3g)$  with  $\text{ind}(S) = k$ , then  $S = (g_1)(g_1)(\bar{x}_2g_1)(\bar{x}_3g_1)$ , where  $g_1 = x_1g$ ,  $1 \leq \bar{x}_2, \bar{x}_3 \leq q-1$ . It follows that

$$k \leq \|S\|_{g_1} = \frac{1 + 1 + \bar{x}_2 + \bar{x}_3}{q},$$

thus  $k < 2$  unless  $\bar{x}_2 = \bar{x}_3 = q-1$ , which is impossible since  $S$  is minimal. So we must have that  $k = 1$ , and Theorem 1.2 is proved.

**Proposition 2.1.** *Let  $G$  be a finite cyclic group, where  $|G| = q$  is a prime power with  $\gcd(q, 6) = 1$ , and let  $S = g((q-a)g)((q-b)g)(cg) \in \mathcal{F}(G)$  be a minimal zero-sum sequence, where  $g \in G$  with  $\text{ord}(g) = q$ ,  $\|S\|_g = 2$  and  $1 < a \leq b < c < q/2$ . Then  $\text{ind}(S) = 1$ .*

We now prove that Theorem 1.2 follows from the above proposition. By [6, Lemma 5.1.2] and the above discussion, we may always assume that  $S = g(x_2g)(x_3g)(x_4g)$  where  $1 < x_2 \leq x_3 \leq x_4 < q$  and  $\text{ind}(S) = k$ . Since  $1 + x_2 + x_3 + x_4 < 3q$ , we can assume that  $k \leq 2$ .

We now note that if all  $x_2, x_3, x_4 > \frac{q}{2}$  and  $1 + x_2 + x_3 + x_4 = 2q$ , we let  $x = q - x_3$  and  $y = q - x_4$ . Then  $1 + x_2 = x + y$ . Thus  $1 < y \leq x < x_2$ , and  $x < \frac{q}{2}$ . Then we obtain  $2 \times 1 = 2$ ,  $|2x_3|_q = q - 2x$ ,  $|2x_4|_q = q - 2y$  and  $|2x_2|_q = 2x_2 - q$  since  $q < 2x_2 < 2q$  where  $|x|_q$  denotes the least positive residue of  $x$  modulo  $q$ . Therefore,  $2 + |2x_2|_q + |2x_3|_q + |2x_4|_q = 2 + 2x_2 - q + q - 2x + q - 2y = 2(1 + x_2 - x - y) + q = q$ , so  $\|S\|_{\frac{(q+1)}{2}g} = 1$  and we are done.

So we may assume that not all  $x_2, x_3, x_4 > \frac{q}{2}$ . Note that if  $x_3 < \frac{q}{2}$ , then  $\|S\|_g < 2$ . So we may assume that  $x_2 < \frac{q}{2}$  and  $x_3, x_4 > \frac{q}{2}$ . We handle such situations as follows. Let  $c = x_2$ ,  $b = q - x_3$ , and  $a = q - x_4$  so that  $a \leq b$ . Then note that  $1 - a - b + c = 0$ , and therefore,  $1 + c = a + b$ , so we must have that  $c > b$ . Then  $\|S\|_g = (1 + c + (q - a) + (q - b))/q = 2$ , and thus Theorem 1.2 follows immediately from Proposition 2.1.

Note that if  $q$  is a prime power and  $m, k, c$  are positive integers such that  $\gcd(q, m) = 1$ ,  $1 \leq m < q$  and  $\lceil \frac{kq}{c} \rceil = m$ ,  $1 \leq k < c < q$ , then  $\frac{kq}{c} < m$ . This fact will be frequently used later in the paper. We also note that if  $k, m$  are positive integers such that

$$\left\lceil \frac{kq}{c} \right\rceil = m < \frac{kq}{b}, \quad \gcd(m, q) = 1, \quad 1 \leq k \leq b, \quad \text{and} \quad ma < q. \quad (2.1)$$

Then  $m + |mc|_q + |m(q-a)|_q + |m(q-b)|_q \leq m + q - ma + (kq - mb) + (mc - kq) = q$ , so  $\text{ind}(S) \leq (m + |mc|_q + |m(q-a)|_q + |m(q-b)|_q)/q = 1$  and then Proposition 2.1 is proved. Therefore, to prove Proposition 2.1, it suffices to choose integers  $k, m$  such that the conditions of (2.1) are satisfied.

## 3. Prime case

In this section, we handle the case when  $q = p$  is prime. As mentioned in the last section, we need only prove Proposition 2.1 for this case. By the remark at the end of the previous section it suffices to find an integer  $m$  with  $\gcd(m, p) = 1$  such  $|m|_p + |mc|_p + |m(p-a)|_p + |m(p-b)|_p = p$ . We do so by proving the following result. Since  $q = p$  is a prime, we may assume that  $a, b$  and  $c$  are distinct in the following proposition.

**Proposition 3.1.** Assume that  $1 + c + (p - a) + (p - b) = 2p$  where  $p$  is a prime and  $1 < a < b < c < \frac{p}{2}$ . Let  $k$  be the smallest positive integer such that  $\lceil \frac{kp}{c} \rceil = m < \frac{kp}{b}$ . Then  $|m|_p + |mc|_p + |m(p - a)|_p + |m(p - b)|_p = p$ .

**Proof.** Note that such an integer  $k$  always exists. Since  $\frac{b}{c}(p) < p - 1$ , we have  $\lceil \frac{b}{c}p \rceil = m \leq p - 1 < p = \frac{b}{b}(p)$ . Thus such a  $k$  exists and  $k \leq b$ . By using the minimality of  $k$  and the fact that  $\frac{kp}{c}$  is not an integer, we can show that  $(k - 1)p < mb < kp$  and  $kp < mc < kp + p$ , so  $|mc|_p = mc - kp$  and  $|m(p - b)|_p = kp - mb$ . If  $ma < p$ , then the conditions of (2.1) are satisfied. In fact, we have that  $|m|_p + |mc|_p + |m(p - a)|_p + |m(p - b)|_p = m + mc - kp + p - ma + kp - mb = p + m(1 + c - a + b) = p$ , and we are done. We now show that  $(*)$  always holds by using a case by case analysis.

**Case 1.** If  $k = 1$ , then  $m < \frac{p}{b} < \frac{p}{a}$ , and therefore  $ma < p$  and we are done.

We remark that if  $k \geq 2$ , then by the minimality of  $k$ ,  $\lceil \frac{(k-1)p}{c} \rceil = \lceil \frac{(k-1)p}{b} \rceil$ . Thus,

$$\frac{(k-1)p}{b} - \frac{(k-1)p}{c} = \frac{(c-b)(k-1)p}{cb} < 1. \quad (3.1)$$

**Case 2.** If  $k = 2$ , then by (3.1), we have  $\frac{(c-b)p}{cb} < 1$ .

**Subcase 1.** If  $a \leq \frac{b}{k} = \frac{b}{2}$ , then  $am < \frac{b}{k} \frac{k}{b} p = p$  and we are done.

**Subcase 2.** If  $a > \frac{b}{k} = \frac{b}{2}$  then if  $b = 2l$ , then  $a > l$  and  $a - 1 \geq l$ . Now  $\frac{(k-1)(c-b)p}{bc} = \frac{a-1}{bc} p = \frac{lp}{bc} > \frac{2l}{b} = 1$ , which is a contradiction to (3.1). If  $b = 2l + 1$ , as before, if  $a - 1 \geq \frac{b}{2}$  we find a contradiction to (3.1). So assume that  $\frac{b}{2} > a - 1 > \frac{b}{2} - 1$ , and thus  $a = l + 1$ . Now  $c = a + b - 1 = 3l + 1$ . Thus  $c - b = a - 1 = l$  and again we have  $\frac{(k-1)(c-b)p}{bc} = \frac{lp}{(2l+1)(c)} \geq \frac{3l}{2l+1} \geq 1$  if  $p \geq 3c$ , which is a contradiction to (3.1). So assume that  $p = 2c + l_0$ , for some  $l_0$  odd. If  $l_0 = 1$  then  $p = 2c + 1 = 2(3l + 1) + 1 = 6l + 3 = 3(2l + 1)$  a contradiction since  $p$  is not divisible by 3. If  $l_0 = 3$  then  $p = 2c + 3$  and thus  $\lceil \frac{p}{c} \rceil = \lceil 2 + \frac{3}{c} \rceil = 3 < \frac{p}{b} = \frac{6l+5}{2l+1}$ . This implies that  $k = 1$  a contradiction. If  $l_0 \geq 5$  then  $p \geq 2c + 5 = 6l + 7$ . Now  $\frac{lp}{(2l+1)(3l+1)} \geq \frac{l(6l+7)}{6l^2+5l+1} > 1$  a contradiction to (3.1).

**Case 3.** If  $k \geq 3$ . As before, if  $a \leq \frac{b}{k}$  then  $am < a \frac{kp}{b} \leq p$  and we are done.

If  $a - 1 \geq \frac{b}{k}$  then  $\frac{(k-1)p(c-b)}{bc} \geq \frac{(k-1)bp}{kbc} > \frac{2(k-1)}{k} > 1$ , a contradiction.

Thus, assume that  $\frac{b}{k} + 1 > a > \frac{b}{k}$ . Assume also that  $b = kl + k_0$  for some  $1 \leq k_0 < k$  and  $l \geq 1$ . Note that if  $k_0 = 0$  then  $a > \frac{b}{k}$ . So  $a > l$ , and thus  $a - 1 \geq l = \frac{b}{k}$ , a contradiction. Then  $a = l + 1$  and  $c = a + b - 1 = (k + 1)l + k_0$  and also  $c - b = a - 1 = l$ . Now, (3.1) reduces to the following:

$$\frac{(c-b)(k-1)p}{cb} = \frac{(k-1)lp}{(kl+k_0)c} < 1. \quad (3.2)$$

If  $l \geq 2$  then we have  $\frac{(c-b)(k-1)p}{cb} \geq \frac{2(k-1)l}{kl+k-1} = \frac{2l-2l}{k(l+1)-1} = 1 + \frac{k(l-1)-2l+1}{k(l+1)-1} \geq 1$  (the first inequality holds since  $\frac{p}{c} > 2$ ,  $k - 1 \geq k_0$  and the second holds since  $k(l - 1) - 2l + 1 \geq 3(l - 1) - 2l + 1 = l - 2 \geq 0$ ), which is a contradiction to (3.2).

If, on the other hand,  $l = 1$ , if  $\frac{p}{c} > 3$  then  $\frac{(k-1)lp}{(kl+k_0)c} > \frac{3(k-1)l}{kl+k-1} = \frac{3k-3}{2k-1} = 1 + \frac{k-2}{2k-1} > 1$ , again a contradiction to (3.2).

So  $p = 2c + s_0$  for some  $c > s_0 \geq 1$ , where  $s_0$  is odd. Recall that  $b = lk + k_0 = k + k_0$ , so  $a = 2$  and  $c = b + 1 = k + k_0 + 1$ .

If  $s_0 \geq 3$ , then  $(k-1)\frac{s_0}{c} - 1 \geq \frac{3k-3}{2k} - 1 = \frac{2k+k-3}{2k} - 1 = k-3 \geq 0$ . Note that  $\frac{(k-1)(c-b)p}{bc} = \frac{(k-1)(2c+s_0)}{(k+k_0)(k+k_0+1)} \geq \frac{(k-1)(2+\frac{s_0}{c})}{2k-1} = 1 + \frac{(k-1)\frac{s_0}{c}-1}{2k-1} \geq 1$ , a contradiction to (3.2), so we must have  $s_0 \leq 2$ . Since  $s_0$  is odd, we must have  $s_0 = 1$ .

If  $s_0 = 1$ , then  $\frac{p}{c} = 2 + \frac{1}{k+k_0+1} < \frac{p}{b} = 2 + \frac{3}{k+k_0}$ . Now let  $X = \lceil \frac{k+k_0}{3} \rceil$ . Then  $X < \frac{k+k_0}{3} + 1 = \frac{k+k_0+3}{3} < k < b$  ( $k \geq 3$ ). We claim that  $X\frac{p}{b}$  is not an integer. Otherwise (since  $p$  is a prime,  $b < p$ )  $\frac{X}{b}$  is an integer, which means  $b \mid X$  but  $X < b$ , a contradiction.

Next consider  $\lceil X\frac{p}{c} \rceil = 2X + \lceil X\frac{1}{k+k_0+1} \rceil = 2X + 1$  (since  $\frac{X}{k+k_0+1} = \frac{X}{c} < 1$ ) and  $X\frac{p}{b} = 2X + \frac{3X}{k+k_0} \geq 2X + 3\frac{k+k_0}{3} \frac{1}{k+k_0} \geq 2X + 1$ . Since  $X\frac{p}{b}$  is not an integer, we have that  $X\frac{p}{b} > 2X + 1 = \lceil X\frac{p}{c} \rceil$ . By the minimality of  $k$ ,  $X \geq k$  contradictory to  $X < k$ . In all cases we showed that  $ma < p$  must hold and the proof is complete.  $\square$

#### 4. Four lemmas

In this section, we establish four useful lemmas. We will prove that Proposition 2.1 holds for  $a = 2, 3, 4$  in Lemmas 4.1, 4.2, 4.3 respectively, and it holds for  $q = 5^l$  and  $4 < \frac{2q}{c} < 5 < \frac{2q}{b} < 6$  in Lemma 4.4. Since the prime case was handled in last section, from now on we may always assume that  $q = p^n$  with  $n \geq 2$  is prime power, but not a prime. Since  $\gcd(q, 6) = 1$ ,  $q$  is at least 25.

**Lemma 4.1.** Proposition 2.1 holds for  $a = 2$ .

**Proof.** Since  $a = 2$ , we have  $\{1, q-a, q-b, c\} = \{1, q-2, q-b, b+1\}$ . If  $b = 2t$ , then let  $m = (q-1)/2$ . Clearly  $(m, q) = 1$  and  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |mc|_q\} = \{m, 1, t, q-m-t-1\}$ , so  $|m|_q + |m(q-a)|_q + |m(q-b)|_q + |mc|_q = q$  and then  $\text{ind}(S) = 1$ .

If  $b = 2t+1$ , let  $m = (q-1)/2$ . Then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |mc|_q\} = \{m, 1, q - \frac{q-1-2t}{2}, q - (t+1)\} = \{1, \frac{q-1}{2}, q - (t+1), q - \frac{q-b}{2}\} = \{1, c_1, q-a_1, q-b_1\}$ , where  $a_1 = (t+1) \leq b_1 = (q-b)/2 < c_1 = (q-1)/2 < q/2$ . Next we will find constants  $k_1$  and  $m_1$  satisfying (2.1) and we are done by the remark at the end of Section 2. Note that

$$lq = 2l \cdot \frac{q-1}{2} + l, \quad lq = 2q \cdot \frac{q-b}{2} + lb, \quad \text{for all } l = 1, 2, \dots, \frac{q-5}{2}.$$

Take  $k = \lceil \frac{q-b}{2b} \rceil$ . Then we have

$$2k < \left\lceil \frac{kq}{(q-1)/2} \right\rceil = 2k+1 < \frac{kq}{(q-b)/2},$$

$$2(k+1) < \left\lceil \frac{(k+1)q}{(q-1)/2} \right\rceil = 2k+3 < \frac{(k+1)q}{(q-b)/2}.$$

Observe that either  $\gcd(2k+1, q) = 1$  or  $\gcd(2k+3, q) = 1$ . If  $\gcd(2k+1, q) = 1$ , take  $k_1 = k$  and then  $m_1 = \lceil \frac{kq}{(q-1)/2} \rceil = 2k+1$ ; otherwise, take  $k_1 = k+1$  and then  $m_1 = 2k+3$ . The above inequalities show that  $\lceil \frac{k_1 q}{c} \rceil = m_1 < \frac{k_1 q}{b}$ ,  $\gcd(m_1, q) = 1$ ,  $1 \leq k_1 \leq b_1$ . To verify the conditions of (2.1), we need only show that  $m_1 a_1 < q$ . Since  $m_1 \leq 2k+3$  and  $a_1 = t+1$  it suffices to show that  $(2k+3)(t+1) < q$ .

Let  $q = rb + b_0$ . If  $r = 1$ , then take  $k_1 = 1$  and  $m_1 = 2$ . Thus  $m_1 a_1 = 2(t+1) < 2t+1+b_0 = q$  as desired. Next assume that  $r \geq 2$ . If  $r$  is odd, then  $k = \lceil \frac{q-b}{2b} \rceil = \frac{r+1}{2}$ , so we have  $(2k+3)(t+1) = (r+4)(t+1) < r(2t+1) + b_0 = q$  unless  $r = 3$  or  $r = 5$ ,  $t \leq 4$  or  $r = 7$ ,  $t = 1$ . If  $r = 5$ ,  $t \leq 4$  or  $r = 7$ ,  $t = 1$ , and  $q \leq (r+4)(t+1) \leq \max\{9 \times 5, 11 \times 2\} = 45$ , then  $q = 25$  since  $\gcd(q, 6) = 1$  and  $q$  is a prime power (but not a prime). If  $r = 5$  and  $q = 25$ , then take  $k_1 = 3$  and  $m_1 = 7$ . Thus  $\gcd(m_1, q) = 1$  and  $m_1 a_1 = 7(t+1) < 5(2t+1) + b_0 = 25$  as desired. If  $r = 7$  and  $q = 25$ , then take  $k_1 = 1$  and

$m_1 = 3$ . Thus  $\gcd(m_1, q) = 1$  and  $m_1 a_1 = 3(t+1) = 6 < 7(2t+1) + b_0 = 25$  as desired. If  $r = 3$  and  $5 \nmid q$ , then  $k_1 = 2$  and  $5(t+1) < 3(2t+1) + b_0 < q$ . If  $r = 3$ ,  $5 \mid q$  and  $7(t+1) > q = 6t + 3 + b_0$ , then we have  $b_0 \leq t + 4$ . In this case, we take  $m = \frac{q-11}{2}$ , then  $\{m, |m(q-2)|_q, |m(q-b)|_q, |m(b+1)|_q\} = \{m, 11, 2q - 11(t+1), 11t - m - q\}$  which has sum  $q$  and we are done.

If  $r$  is even, then  $k = \lceil \frac{q-b}{2b} \rceil = \frac{r}{2}$ , so we have  $(2k+3)(t+1) = (r+2)(t+1) < r(2t+1) + b_0 = q$  unless  $r = 2$ . If  $r = 2$ , then  $k = 1$ . Let  $k_1 = k = 1$  and thus  $m_1 = 3$ . Then  $\gcd(m_1, q) = 1$  and  $m_1 a_1 = (2k+1)(t+1) = 3(t+1) < 2(2t+1) + b_0 = q$  as desired.

In all cases we showed that Proposition 2.1 holds and the proof is complete.  $\square$

**Lemma 4.2.** Proposition 2.1 holds for  $a = 3$ .

**Proof.** If  $a = 3$ , then we have  $\{1, q-a, q-b, c\} = \{1, q-3, q-b, b+2\}$ . We divide the proof into the following four cases.

**Case 1.**  $q \equiv 1 \pmod{3}$  and  $b = 3t$  or  $3t+2$ .

(i) If  $b = 3t$ , we take  $m = \frac{q-1}{3}$ . Then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 1, t, q-t-m-1\}$ . Since  $|m|_q + |m(q-a)|_q + |m(q-b)|_q + |m(q-c)|_q = q$ ,  $\text{ind}(S) = 1$ .

(ii) If  $b = 3t+2$ , we take  $m = \frac{q-1}{3}$ . Then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{1, m, t+m+1, m-t-1\}$ . It follows that  $|m|_q + |m(q-a)|_q + |m(q-b)|_q + |m(q-c)|_q = q$  and thus  $\text{ind}(S) = 1$ .

**Case 2.**  $q \equiv 2 \pmod{3}$  and  $b = 3t$  or  $b = 3t+2$ .

(i) If  $b = 3t$ , we take  $m = \frac{2q-1}{3}$ . Then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{1, q - \frac{q+1}{3}, t, \frac{q-2}{3} - t\}$ . Since the sum of the new sequence is  $q$ ,  $\text{ind}(S) = 1$ .

(ii) If  $b = 3t+2$ , we take  $m = \frac{q-2}{3}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 2, 2t+2=m, m-2(t+1)\}$ . It follows that  $|m|_q + |m(q-a)|_q + |m(q-b)|_q + |m(q-c)|_q = q$  and thus  $\text{ind}(S) = 1$ .

**Case 3.**  $q \equiv 2 \pmod{3}$  and  $b = 3t+1$ .

Let  $m = \frac{2q-1}{3}$ . Then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{1, q - \frac{q+1}{3}, \frac{q+b}{3}, q-t-1\}$ . Now we have  $q+1 \geq 3(t+1)$  and

$$lq = 3l \cdot \frac{q+1}{3} - l, \quad lq = 3l \cdot \frac{q+b}{3} - lb, \quad \text{for all } l = 1, 2, \dots, \frac{q-5}{3}.$$

Take  $k = \lceil \frac{q+b}{3b} \rceil$ . Then we have

$$\begin{aligned} 3k-2 &< \left\lceil \frac{kq}{(q+b)/3} \right\rceil = 3k-1 < \frac{kq}{(q+1)/3}, \\ 3k+1 &< \left\lceil \frac{(k+1)q}{(q+b)/3} \right\rceil = 3k+2 < \frac{(k+1)q}{(q+1)/3}, \end{aligned}$$

or

$$3k < \left\lceil \frac{(k+1)q}{(q+b)/3} \right\rceil < 3k+2 < \frac{(k+1)q}{(q+1)/3}.$$

Note that either  $\gcd(3k-1, q) = 1$  or  $\gcd(3k+2, q) = 1$ . As in the proof of Lemma 4.1, if  $\gcd(3k-1, q) = 1$ , we take  $k_1 = k$  and thus  $m_1 = 3k-1$ ; otherwise, we take  $k_1 = k+1$  and thus  $m_1 = 3k+1$  or  $m_1 = 3k+2$ . To prove this case, it suffices to show that  $m_1(t+1) < q$ .

Let  $q = rb + b_0$ . If  $r = 3s + 1$ , then  $k = \lceil \frac{q+b}{3b} \rceil = \frac{r+2}{3}$ , so  $m_1(t+1) \leq (3k+2)(t+1) = (r+4)(t+1) < r(3t+1) + b_0 = q$ . If  $r = 3s + 2$ , then  $k = \lceil \frac{q+b}{3b} \rceil = \frac{r+4}{3}$ , so  $(3k+2)(t+1) = (r+6)(t+1) < r(3t+1) + b_0 = q$  holds unless  $r = 2$  or  $r = 5$  and  $t = 1$ . If  $r = 2$ , then  $k = 1$  and  $m_1 = 2$  with  $\gcd(2, q) = 1$ . Thus  $m_1(t+1) = 2(t+1) < 2(3t+1) + b_0 = q$ . If  $r = 5$ ,  $t = 1$ , then  $q < 6 \times 4 < 25$  which is impossible.

If  $r = 3s$ , then  $k = \lceil \frac{q+b}{3b} \rceil = \frac{r+3}{3}$ , so  $(3k+2)(t+1) = (r+5)(t+1) < r(3t+1) + b_0 = q$  holds unless  $r = 3$  and  $t \leq 5$ . If  $r = 3$ ,  $t \leq 5$ , then  $k = 2$  and  $m_1 = 3k + 2 = 8$ . Now  $8(t+1) \leq 3(3t+1) + b_0$  does not hold only when  $q \leq 48$ . This gives that  $q = 25 \not\equiv 2 \pmod{3}$ , which yields a contradiction to the assumption.

**Case 4.**  $q \equiv 1 \pmod{3}$  and  $b = 3t + 1$ .

Take  $m = \frac{q-1}{3}$ . Then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{1, \frac{q-1}{3}, q - \frac{q-b}{3}, q - t - 1\}$ . Now we have  $q - b \geq 3(t+1)$  and

$$lq = 3l \cdot \frac{q-1}{3} + l, \quad lq = 3l \cdot \frac{q-b}{3} + lb, \quad \text{for all } l = 1, 2, \dots, \frac{q-5}{3}.$$

Take  $k = \lceil \frac{q-b}{3b} \rceil$ , then we have

$$3k < \left\lceil \frac{kq}{(q-1)/3} \right\rceil = 3k + 1 < \frac{kq}{(q-b)/3},$$

$$3(k+1) < \left\lceil \frac{(k+1)q}{(q-1)/3} \right\rceil = 3k + 4 < \frac{(k+1)q}{(q-b)/3}.$$

Note that either  $\gcd(3k+1, q) = 1$  or  $(3k+4, q) = 1$ . Let  $q = rb + b_0$ . If  $r = 3s + 1$ , then  $k = \lceil \frac{q-b}{3b} \rceil = \frac{r+2}{3}$ ,  $(3k+4)(t+1) = (r+6)(t+1) \leq r(3t+1) + b_0$  holds unless  $t \leq 2$  and  $r = 4$ . If  $r = 4$  and  $t \leq 2$ , then  $q = 25$  since  $q$  is a prime power and  $\gcd(q, 6) = 1$ . Now  $k = 2$  and  $m_1 = 3k + 1 = 7$  with  $\gcd(7, 25) = 1$ . Thus  $7(t+1) < 25$  as desired.

If  $r = 3s + 2$ , then  $k = \lceil \frac{q-b}{3b} \rceil = \frac{r+1}{3}$ , so  $(3k+4)(t+1) = (r+5)(t+1) \leq r(3t+1) + b_0$  holds unless  $r = 2$ . If  $r = 2$ , then  $k = 1$ ,  $\gcd(4, q) = 1$  and  $4(t+1) < 2(3t+1) + b_0$  as desired.

If  $r = 3s$ , then  $k = \lceil \frac{q-b}{3b} \rceil = \frac{r}{3}$ ,  $(3k+4)(t+1) = (r+4)(t+1) \leq r(3t+1) + b_0$  holds unless  $r = 3$  and  $t = 1$ . If  $r = 3$  and  $t = 1$ , then  $q < 4 \times 4 < 25$  which is impossible.

In all cases we showed that Proposition 2.1 holds and the proof is complete.  $\square$

**Lemma 4.3.** Proposition 2.1 holds for  $a = 4$ .

**Proof.** If  $a = 4$ , then we have  $\{1, q - a, q - b, c\} = \{1, q - 4, q - b, b + 3\}$ . We divide the proof into the following four cases.

**Case 1.**  $q \equiv 1 \pmod{4}$  and  $b \not\equiv 1 \pmod{4}$ .

(i) If  $b = 4t$ , we take  $m = \frac{q-1}{4}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 1, t, q - t - m - 1\}$ .

(ii) If  $b = 4t + 2$ , we take  $m = \frac{q-1}{4}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 1, t + (q+1)/2, m - t - 1\}$ .

(iii) If  $b = 4t + 3$ , we take  $m = \frac{q-1}{4}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 1, q - 3m + t, 2m - t - 1\}$ .

It follows that in all three subcases  $|m|_q + |m(q-a)|_q + |m(q-b)|_q + |m(q-c)|_q = q$  and thus  $\text{ind}(S) = 1$  as desired.

**Case 2.**  $q \equiv 3 \pmod{4}$  and  $b \not\equiv 1 \pmod{4}$ .

(i) If  $b = 4t$ , we take  $m = \frac{q-3}{4}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 3, 3t, q-3t-m-3\}$ .

(ii) If  $b = 4t+2$ , we take  $m = \frac{q-1}{2}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 1, 2t+1, q-m-2t-2\}$ .

(iii) If  $b = 4t+3$ , we take  $m = \frac{q-3}{4}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{m, 3, 3t+\frac{q+27}{4}, 2m-3t-3\}$ .

It follows that in all these subcases  $|m|_q + |m(q-a)|_q + |m(q-b)|_q + |mc|_q = q$  and thus  $\text{ind}(S) = 1$  as desired.

**Case 3.**  $q \equiv 1 \pmod{4}$  and  $b = 4t+1$ .

Take  $m = \frac{q-1}{4}$ , and then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{(q-1)/4, 1, q-(q-b)/4, q-t-1\}$ . Now we have  $q-b > 4t+4$  and

$$lq = 4l \cdot \frac{q-1}{4} + l, \quad lq = 4l \cdot \frac{q-b}{4} + lb, \quad \text{for all } l = 1, 2, \dots, \frac{q-5}{4}.$$

Take  $k = \lceil \frac{q-b}{4b} \rceil$ . Then we have

$$4k < \left\lceil \frac{kq}{(q-1)/4} \right\rceil = 4k+1 \leq \frac{kq}{(q-b)/4},$$

$$4(k+1) < \left\lceil \frac{(k+1)q}{(q-1)/4} \right\rceil = 4k+5 < \frac{(k+1)q}{(q-b)/4}.$$

Note that either  $\gcd(4k+1, q) = 1$  or  $\gcd(4k+5, q) = 1$ . Let  $q = rb + b_0$ . If  $r = 4s+1$ , then  $k = \lceil \frac{q-b}{2b} \rceil = \frac{r+3}{4}$ ,  $(4k+5)(t+1) = (r+8)(t+1) < r(4t+1) + b_0$  holds unless  $t = 1$  and  $r = 5$ . If  $r = 5$  and  $t = 1$ , then  $q = 25$ , so  $\{1, q-a, q-b, c\} = \{1, 8, 20, 21\}$ . Now  $\{|4|_q, |4(q-a)|_q, |4(q-b)|_q, |4c|_q\} = \{4, 5, 7, 9\}$  and we are done.

If  $r = 4s+2$ , then  $k = \lceil \frac{q-b}{4b} \rceil = \frac{r+2}{4}$ ,  $(4k+5)(t+1) = (r+7)(t+1) < r(4t+1) + b_0$  holds unless  $r = 2$ . If  $r = 2$ , then  $k = 1$ . We first consider the case when  $6 < \frac{kb}{(q-b)/2}$ . Now we have  $6(t+1) < 2(4t+1) \leq q$  and we are done. Next we consider the case when  $6 > \frac{kb}{(q-b)/2}$ . It follows that  $q > 3b$  and so  $9(t+1) < 3(4t+1) < q$  as desired.

If  $r = 4s+3$ , then  $k = \lceil \frac{q-b}{4b} \rceil = \frac{r+1}{4}$ ,  $(4k+5)(t+1) = (r+6)(t+1) < r(4t+1) + b_0$  holds unless  $r = 3$  and  $t \leq 1$ . If  $r = 3$  and  $t \leq 1$ , then  $(r+6)(t+1) \leq 18$ . Since  $q \geq 25$ , we have  $(r+6)(t+1) < q$  as desired.

If  $r = 4s$ , then  $k = \lceil \frac{q-b}{4b} \rceil = \frac{r}{4}$ , and  $(4k+5)(t+1) = (r+5)(t+1) < r(4t+1) + b_0$  as desired.

**Case 4.**  $q \equiv 3 \pmod{4}$  and  $b = 4t+1$ . We take  $m = \frac{3q-1}{4}$ , then  $\{|m|_q, |m(q-a)|_q, |m(q-b)|_q, |m(q-c)|_q\} = \{q-(q+1)/4, 1, q-(t+1), (q+b)/4\}$ . Now we have  $q+1 > 4t+4$  and

$$lq = 4l \cdot \frac{q+1}{4} - l, \quad lq = 4l \cdot \frac{q+b}{4} - lb, \quad \text{for all } l = 1, 2, \dots, \frac{q-5}{4}.$$

Take  $k = \lceil \frac{q+b}{4b} \rceil$ . Then we have

$$4k-2 < \left\lceil \frac{kq}{(q+b)/4} \right\rceil = 4k-1 \leq \frac{kq}{(q+1)/4},$$

$$4(k+1)-2 < \left\lceil \frac{(k+1)q}{(q+b)/4} \right\rceil = 4k+3 < \frac{(k+1)q}{(q+1)/4},$$



or

$$4(k+1) - 3 < \left\lceil \frac{(k+1)q}{(q+b)/4} \right\rceil < 4k+3 < \frac{(k+1)q}{(q+1)/4}.$$

Note that either  $\gcd(4k-1, q) = 1$  or  $\gcd(4k+3, q) = 1$ . Let  $q = rb + b_0$ . If  $r = 4s+1$ , then  $k = \lceil \frac{q+b}{4b} \rceil = \frac{r+3}{4}$ , so  $(4k+3)(t+1) = (r+6)(t+1) \leq r(4t+1) + b_0$  holds unless  $r = 5$  and  $t = 1$ . If  $r = 4s+2$ , then  $k = \lceil \frac{q+b}{4b} \rceil = \frac{r+2}{4}$ , so  $(4k+3)(t+1) = (r+5)(t+1) < r(4t+1) + b_0$  holds unless  $r = 2$  and  $t \leq 5$ . If  $r = 4s+3$ , then  $k = \lceil \frac{q+b}{4b} \rceil = \frac{r+5}{4}$ ,  $(4k+3)(t+1) = (r+8)(t+1) < r(4t+1) + b_0$  holds unless  $r = 3$  and  $t \leq 7$ . If  $r = 4s$ , then  $k = \lceil \frac{q+b}{4b} \rceil = \frac{r+4}{4}$ , so  $(4k+3)(t+1) = (r+7)(t+1) < r(4t+1) + b_0$  holds unless  $r = 4$  and  $t = 1$ . If  $r = 5$ ,  $t = 1$ , or  $r = 4$ ,  $t = 1$ , or  $r = 2$ ,  $t \leq 5$ , or  $r = 3$ ,  $t \leq 7$ , then  $q < \max\{4 \times (4 \times 7 + 1), 6 \times (4 + 1), 3 \times (4 \times 5 + 1)\} = 116$ . Thus  $q = 25$  or  $49$  since  $q$  is a prime power (but not a prime), and  $\gcd(q, 6) = 1$ , which contradicts the assumption  $q \equiv 3 \pmod{4}$ .

This completes the proof.  $\square$

**Lemma 4.4.** Let  $q = 5^k$ ,  $k \geq 2$  and  $b, c$  positive integers with  $4 < \frac{2q}{c} < 5 < \frac{2q}{b} < 6$ . Then Proposition 2.1 holds.

**Proof.** Since  $6 < \frac{3q}{c} < \frac{3q}{b} < 9$ , we distinguish three cases.

**Case 1.**  $6 < \frac{3q}{c} < 7 < \frac{3q}{b} < 8$ . Then  $b > \frac{3q}{7}$ , and thus

$$a = c - b + 1 \leq \frac{q-1}{2} - \frac{3q+1}{7} + 1 = \frac{q+5}{14}.$$

It follows that  $7a \leq \frac{q+5}{2} < q$  and we are done.

**Case 2.**  $7 < \frac{3q}{c} < 8 < \frac{3q}{b} < 9$ . Then  $c < \frac{3q}{7}$  and  $b > \frac{3q}{8}$ , and thus

$$a = c - b + 1 \leq \frac{3q-1}{7} - \frac{3q+1}{8} + 1 = \frac{3q+41}{56}.$$

In this case, we have  $8a \leq \frac{3q+41}{7} < q$  and we are done.

**Case 3.**  $6 < \frac{3q}{c} < 7 < 8 < \frac{3q}{b} < 9$ . Then  $b > \frac{3q}{8}$ , and thus

$$a = c - b + 1 \leq \frac{q-1}{2} - \frac{3q+1}{8} + 1 = \frac{q+3}{8}.$$

Since  $q \geq 25$ , we obtain  $7a \leq \frac{7q+21}{8} < q$  as desired.

Lemma 4.4 is proved.  $\square$

## 5. Proof of Proposition 2.1

We now complete the proof of Proposition 2.1.

**Proof.** By Lemmas 4.1, 4.2, and 4.3, we may assume that  $a > 4$ . Let  $k$  be the largest positive integer such that  $\lceil \frac{(k-1)q}{c} \rceil = \lceil \frac{(k-1)q}{b} \rceil$  and  $\lceil \frac{kq}{c} \rceil = m < \frac{kq}{b}$ .

Note that such an integer  $k$  always exists. Since  $\frac{bq}{c} < q-1$ , we have that  $\lceil \frac{bq}{c} \rceil = m \leq q-1 < q = \frac{bq}{b}$ . Thus such a  $k$  exists and  $k \leq b$ . Next we show that we can always find  $m$  as defined above such that  $\gcd(m, q) = 1$  and  $ma < q$ , so the conditions of (2.1) are satisfied and we are done.

**Case 1.**  $k = 1$ . Then  $\lceil \frac{q}{c} \rceil = m < \frac{q}{b}$ . If  $\gcd(m, q) = 1$ , then  $m < \frac{q}{b} < \frac{q}{a}$  and therefore  $ma < q$  as desired.

If  $\gcd(m, q) > 1$  and  $\lceil \frac{q}{c} \rceil = m < m + 1 \leq \frac{q}{b}$ , then we have  $(m + 1) < \frac{q}{b} < \frac{q}{a}$ . Thus  $(m + 1)a < q$  and  $\gcd(m + 1, q) = 1$  since  $q$  is a prime power and  $\gcd(m, q) > 1$ .

Next we may assume that  $\gcd(m, q) = m_1 > 1$ ,  $q = m_1 q_1$ ,  $m = m_1 r$  and

$$m_1 r - 1 < \frac{m_1 q_1}{c} \leq m_1 r < \frac{m_1 q_1}{b} < m_1 r + 1.$$

This implies that  $q > 5b$  since  $\gcd(q, 6) = 1$ . Since

$$\frac{lm_1 q_1}{c} \leq lm_1 r < \frac{lm_1 q_1}{b} \quad (5.1)$$

holds for any positive integer  $l$ . We take  $l$  to be the largest integer such that

$$lm_1 r - 1 < \frac{lm_1 q_1}{c} \leq lm_1 r < \frac{lm_1 q_1}{b} < lm_1 r + 1.$$

It follows that  $lq = lm_1 br + b_0$ ,  $1 \leq b_0 < b$ . Since  $(lm_1 r - 1)(b + a - 1) = (lm_1 r - 1)c < lq = lm_1 br + b_0$ , we obtain

$$(lm_1 r - 1)(a - 1) < 2b. \quad (5.2)$$

By the definition of  $l$  and (5.1), there are at least two integers in the interval  $[\frac{(l+1)q}{c}, \frac{(l+1)q}{b}]$  and  $(l + 1)m_1 r \in [\frac{(l+1)q}{c}, \frac{(l+1)q}{b}]$ . Since  $\gcd(q, [(l + 1)m_1 r - 1][(l + 1)m_1 r + 1]) = 1$ , as mentioned earlier it suffices to show that either  $[(l + 1)m_1 r + 1]a < q$  or  $[(l + 1)m_1 r - 1]a < q$  when  $\frac{(l+1)q}{c} < (l + 1)m_1 r - 1$ .

Suppose  $l > 1$ . Since

$$\frac{[(l + 1)m_1 r + 1]a}{(lm_1 r - 1)(a - 1)} \leq \frac{5}{4} \cdot \frac{(2 + 1)m_1 r + 1}{2m_1 r - 1} < \frac{5}{2},$$

we have  $[(l + 1)m_1 r + 1]a < 5(lm_1 r - 1)(a - 1)/2 < 5b < q$  as desired.

If  $l = 1$  and  $5 \nmid q$  or  $l = 1$ ,  $q > 6b$ , then we have  $q > 6b$ . Since

$$\frac{[(l + 1)m_1 r + 1]a}{(lm_1 r - 1)(a - 1)} \leq \frac{5}{4} \cdot \frac{2m_1 r + 1}{m_1 r - 1} < 3,$$

we have  $[(l + 1)m_1 r + 1]a < 3(lm_1 r - 1)(a - 1) < 6b < q$  as desired.

If  $l = 1$ ,  $5 \mid q$ , and  $q < 6b$ , then

$$4 < \left\lceil \frac{q}{c} \right\rceil = 5 < \frac{q}{b} < 6,$$

implying that  $4(a - 1) < 2b$  and  $q > 4c \geq 4(5 + 4) = 36$  since  $b \geq a \geq 5$ . Since  $5 \mid q$ , we have  $q \geq 125$ . If  $a \leq 11$ , then  $11a \leq 11 \times 11 < 125 \leq q$  and we are done. So we may assume that  $a \geq 12$ . By the definition of  $l$ , we have

$$\frac{2q}{c} < 9, \quad \text{or} \quad 11 < \frac{2q}{b}.$$

If  $\frac{2q}{c} < 9$ , then  $9a \leq \frac{9 \times 12}{4 \times 11} \times 4(a - 1) < \frac{54}{11}b < 5b \leq q$  as desired. If  $9 < \frac{2q}{c}$ , then  $q > \frac{11b}{2}$  and  $9(a - 1) < 4b$ . It follows that

$$11a \leq \frac{11 \times 12}{9 \times 11} \times 9(a-1) < \frac{48}{9}b < \frac{11b}{2} < q$$

and we are done.

We remark that if  $k \geq 2$ , then by the definition of  $k$ ,  $\lceil \frac{(k-1)q}{c} \rceil = \lceil \frac{(k-1)q}{b} \rceil$ . Thus,

$$\frac{(k-1)q}{b} - \frac{(k-1)q}{c} = \frac{(c-b)(k-1)q}{bc} < 1. \quad (5.3)$$

**Case 2.**  $k \geq 2$ .

**Subcase 2.1.** If  $a > b/k$ , then by using similar arguments to Subcase 2 and Case 3 of Proposition 3.1, we can derive a contradiction.

**Subcase 2.2.** If  $a \leq b/k$  and  $\gcd(m, q) = 1$ , then  $am < \frac{b}{k} \cdot \frac{kq}{b} < q$  and we are done. If  $a \leq b/k$ ,  $\gcd(m, q) > 1$  and  $\lceil \frac{q}{c} \rceil = m < m+1 \leq \frac{q}{b}$ , then we have  $(m+1) < \frac{kq}{b}$  and therefore  $(m+1)a < q$ . Note that since  $q$  is a prime power and  $\gcd(m, q) > 1$ ,  $\gcd(m+1, q) = 1$  and we are done.

Next, we may assume that  $\gcd(m, q) = m_1 > 1$ ,  $q = m_1 q_1$ ,  $m = m_1 r$  and

$$m_1 r - 1 < \frac{km_1 q_1}{c} = m_1 r < \frac{km_1 q_1}{b} < m_1 r + 1. \quad (5.4)$$

**Subcase 2.3.** Suppose that  $k \geq 2$  and  $\lceil \frac{(k+1)q}{c} \rceil = \lceil \frac{(k+1)q}{b} \rceil$ . Let  $s$  be the largest positive integer such that

$$\left\lceil \frac{(k+j)q}{c} \right\rceil = \left\lceil \frac{(k+j)q}{b} \right\rceil, \quad \text{for } j = 1, \dots, s.$$

Then  $2 \leq k+s < b$  and

$$\frac{(k+s)(a-1)}{bc} < 1. \quad (5.5)$$

By the definitions of  $k$  and  $s$ , there exists at least one integer  $M$  such that  $\gcd(q, M) = 1$  and

$$\frac{(k+s+1)q}{c} < M < \frac{(k+s+1)q}{b}.$$

(i) If  $a \leq \frac{b}{k+s+1}$ , then  $aM < \frac{b}{k+s+1} \cdot \frac{(k+s+1)q}{b} = q$  and we are done.

(ii) If  $a-1 \geq \frac{b}{k+s+1}$ , then

$$\frac{(k+s)(a-1)q}{bc} \geq \frac{k+s}{k+s+1} \cdot \frac{q}{c} > \frac{2(k+s)}{k+s+1} > 1,$$

which yields a contradiction to (5.5). Hence  $\frac{b}{k+s+1} > a-1 > \frac{b}{k+s+1} - 1$ . Let  $b = (k+s+1)l + k_0$ ,  $1 \leq k_0 \leq k+s$  and  $l \geq a-1 \geq 4$ . Note that if  $k_0 = 0$ , then  $a > \frac{b}{k+s+1}$ , so  $a > l$ , and thus  $a-1 \geq \frac{b}{k+s+1}$  giving a contradiction. Thus  $a = l+1$  and  $c = a+b-1 = (k+s+2)l + k_0$ . Then  $c-b = a-1 = l = a-1 \geq 4$  since  $a \geq 5$ . Now we have

$$\frac{l(k+s)q}{bc} = \frac{l(k+s)q}{((k+s+1)l+k_0)((k+s+2)l+k_0)} > \frac{2(k+s)l}{(k+s+1)l+k+s} \geq 1,$$

which yields a contradiction to (5.5).

**Subcase 2.4.**  $k \geq 3$  with  $\lceil \frac{(k+1)q}{c} \rceil = \lceil \frac{(k+1)q}{c} \rceil$ , and  $4b \geq q$ ,  $5 \mid q$  or  $q \leq 6b$ . By the assumption, there exists an integer  $M$  such that

$$M - 1 \leq \left\lceil \frac{(k+1)m_1q_1}{c} \right\rceil = M < \frac{(k+1)m_1q_1}{b},$$

and  $m < M < m + 5$  ( $5 \mid q$ ) or  $m < M < m + 7$ , so  $\gcd(M, q) = 1$ .

(i) If  $a \leq \frac{b}{k+1}$ , then  $Ma < \frac{b}{k+1} \frac{(k+1)q}{b} = q$  and we are done.

(ii) If  $a - 1 \geq \frac{b}{k+1}$ , then

$$\frac{(k-1)(c-b)q}{bc} = \frac{(k-1)(a-1)q}{bc} \geq \frac{k-1}{k+1} \frac{q}{c} > 1,$$

which yields a contradiction to (5.3). Hence  $\frac{b}{k+1} > a - 1 > \frac{b}{k+1} - 1$ . As before, let  $b = (k+1)l + k_0$ . Then we have  $k_0 \neq 0$ ,  $a = l + 1$  and  $c = a + b - 1 = (k+2)l + k_0$ . If  $k > 3$ , then  $(k-3)(l-1) \geq 3$ , and hence

$$\frac{l(k-1)q}{bc} = \frac{l(k-1)q}{((k+1)l+k_0)((k+2)l+k_0)} > \frac{2(k-1)l}{(k+1)l+k} \geq 1,$$

which yields a contradiction to (5.3). If  $k = 3$ , since  $(m_1r - 1)l < 2b \leq 2(4l + 3) - 1$  we have  $m_1r < 10$  and so  $m_1r = 5, 7$ .

Note that  $m_1r = 5$  is impossible because  $6 < \lceil \frac{3q}{c} \rceil$ . If  $m_1r = 7$ , then

$$6 < \left\lceil \frac{3q}{c} \right\rceil = 7 < \frac{3q}{b} < 8,$$

which implies that  $10l + 3 \leq q \leq 11l + 6$ . If  $k_0 = 3$  and  $9 < \frac{4q}{5l+3}$ , then  $q \geq 11l + 8$ , which is impossible. If  $k_0 = 2$  and  $9 < \frac{4q}{5l+2}$ , then  $q \geq 11l + 6$ , contradicting  $q < \frac{8}{3} \cdot (4l + 2) \leq 11l + 2$ . If  $k_0 = 1$  and  $9 < \frac{4q}{5l+1}$ , then  $q \geq 11l + 4$ , contradicting  $q < \frac{8}{3} \cdot (4l + 1) \leq 11l + 1$ . It follows that  $\lceil \frac{4q}{c} \rceil < 9 < \frac{4q}{b}$ . Note that  $9(l+1) < 10l + 3 \leq q$  when  $l > 6$ . If  $l \leq 6$ , then  $q < 72$ , and therefore,  $q = 49$  and  $l = 4$ . Now  $9(l+1) = 45 < 49$  as desired.

**Subcase 2.5.**  $k \geq 3$  with  $\lceil \frac{(k+1)q}{c} \rceil \neq \lceil \frac{(k+1)q}{c} \rceil$  and  $4b < q < 6b$ ,  $5 \mid q$  or  $q \geq 6b$ . By (5.4), we have  $kq_1 = br + b_0$  for some  $b_0$  with  $1 \leq b_0 < b$ . Let  $l$  be the largest positive integer such that

$$lm_1r - 1 < \left\lceil \frac{lkm_1q_1}{c} \right\rceil = lm_1r < \frac{lkm_1q_1}{b} < lm_1r + 1.$$

As in Case 1, we have  $(lm_1r - 1)(a - 1) < 2b$ . We first consider the case when  $q \geq 6b$ . As before, it suffices to show that  $[(l+1)m_1r + 1]a < q$ .

If  $l > 1$ , then

$$\frac{[(l+1)m_1r + 1]a}{(lm_1r - 1)(a - 1)} \leq \frac{3m_1r + 1}{2m_1r - 1} \cdot \frac{5}{4} < 3,$$

so  $[(l+1)m_1r + 1]a < 3(lm_1r - 1)(a - 1) < 6b \leq q$  as desired.

If  $l = 1$ , then  $m_1r > 3q/b - 1 > 11$ , so

$$\frac{[(l+1)m_1r + 1]a}{(lm_1r - 1)(a - 1)} \leq \frac{2m_1r + 1}{m_1r - 1} \cdot \frac{5}{4} < 3.$$

Hence  $[(l+1)m_1r + 1]a < 3(m_1r - 1)(a - 1) < 6b \leq q$  as desired.

Next we consider the case when  $k \geq 3$ ,  $5 \mid q$  and  $4b < q < 6b$ . Then  $m_1 r > \frac{3q}{b} - 1 > 10$ . Let  $l_1$  be the largest positive integer such that

$$m - 1 < \left\lceil \frac{l_1 m_1 q_1}{c} \right\rceil = m < \frac{l_1 m_1 q_1}{b} < m + 1. \quad (5.6)$$

Then  $m \geq m_1 r \geq 11$  and  $(m - 1)(a - 1) < 2b$ . By the same argument as Subcase 2.3, we can assume that  $\lceil \frac{(l_1+1)m_1 q_1}{c} \rceil < \lceil \frac{(l_1+1)m_1 q_1}{b} \rceil$ .

If  $\gcd(m, 5) = 1$ , then  $ma \leq 2(m - 1)(a - 1) < 4b \leq q$ . If  $5 \mid m$ , then since  $m \geq 11$  we have  $(m + 6)a \leq 2(m - 1)(a - 1) < 4b \leq q$ . Since there exist at least two integers in the interval  $[\frac{(l_1+1)q}{c}, \frac{(l_1+1)q}{b}]$  by the definition of  $l_1$  and  $\frac{(l_1+1)q}{b} < m + 7$ , we can find an  $m_1$  in this interval such that  $\gcd(m_1, 5) = 1$ . Now  $m'a \leq (m + 6)a < q$  as desired.

**Subcase 2.6.**  $k = 2$  and  $\lceil \frac{3q}{c} \rceil < \lceil \frac{3q}{b} \rceil$ . As before, we assume that  $\gcd(m, q) = m_1 > 1$ ,  $q = m_1 q_1$ ,  $m = m_1 r$  and

$$m_1 r - 1 < \frac{2m_1 q_1}{c} \leq m_1 r < \frac{2m_1 q_1}{b} < m_1 r + 1.$$

Since

$$\frac{2lm_1 q_1}{c} \leq lm_1 r_1 < \frac{2lm_1 q_1}{b}$$

holds for any positive integer  $l$ , we can take  $l$  to be the largest integer such that

$$lm_1 r - 1 < \frac{2lm_1 q_1}{c} \leq lm_1 r_1 < \frac{2lm_1 q_1}{b} < lm_1 r + 1.$$

As before, we have

$$(lm_1 r - 1)(a - 1) < 2b. \quad (5.7)$$

(i) If  $m_1 r \geq 12$ , then  $q > 6b$ . Since

$$\frac{[(l + 1)m_1 r + 1]a}{(lm_1 r - 1)(a - 1)} \leq \frac{2 \times 12 + 1}{12 - 1} \times \frac{5}{4} = \frac{125}{44} < 3,$$

we have  $[(l + 1)m_1 r + 1]a < q$  as desired. Therefore  $m_1 r = 5, 7, 10$  or  $11$ .

(ii) If  $l > 1$  and  $m_1 r = 10$ ,  $5 \mid q$  or  $m_1 r = 11$ , then  $q > 5b$ . As in (i), we have

$$\frac{[(l + 1)m_1 r + 1]a}{(lm_1 r - 1)(a - 1)} \leq \frac{3 \times 10 + 1}{2 \times 10 - 1} \times \frac{5}{4} < \frac{5}{2}$$

and we are done.

(iii) If  $l = 1$ ,  $m_1 r = 10$  and  $5 \mid q$ , then

$$9 < \left\lceil \frac{2q}{c} \right\rceil = 10 < \frac{2q}{b} < 11,$$

so  $5b < q < 5.5b$ . If  $b \leq 20$ , then  $q < 110$ , so  $q = 25$ . This implies that  $b \in (\frac{50}{11}, 5)$ , which is impossible. By the definition of  $l$ , we may assume that there exists an integer  $M \leq 21$  such that  $\gcd(q, M) = 1$  and

$$\frac{4q}{c} < M < \frac{4q}{b}.$$

If  $14 < \frac{3q}{b}$ , then  $q > \frac{14}{3}b + \frac{14}{3}(a-1)$ , so  $21a < \frac{14}{3}(a-1) + \frac{49}{3}(\frac{2b}{9} + 1) + \frac{14}{3} \leq \frac{14}{3}b + \frac{14}{3}(a-1) < q$  as desired.

If  $\frac{3q}{c} < 14$ , then  $14 < \frac{3q}{b}$  since  $q > 5b$ . Observe that  $14a < \frac{14}{9} \times \frac{5}{4} \times 9(a-1) < 4b < q$ , as desired.

(iv) If  $l = 1$  and  $m_1r = 11$ , then

$$10 < \frac{2q}{c} < 11 < \frac{2q}{b} < 12,$$

so  $5.5b < q < 6b$ . By the assumptions, there is an integer  $M$  such that

$$\frac{3q}{c} < M < \frac{3q}{b}.$$

Since  $13 \leq M \leq 18$ , we have  $\gcd(q, M) = 1$ . If  $a \leq \frac{b}{3}$ , then  $aM \leq \frac{b}{3} \cdot \frac{3q}{b} = q$  and we are done. If  $a > \frac{b}{3}$ , then

$$\frac{(a-1)q}{bc} \geq \frac{5(b-2)}{b} > 1,$$

which yields a contradiction to (5.3).

(v) If  $l = 1$  and  $m_1r = 7$ , then

$$6 < \frac{2q}{c} < 7 < \frac{2q}{b} < 8,$$

so  $3.5b < q < 4b$ . By the assumptions, there is an integer  $M$  such that

$$\frac{3q}{c} < M < \frac{3q}{b}.$$

Since  $9 \leq M \leq 12$ , we have  $\gcd(q, M) = 1$ . If  $a \leq \frac{b}{3}$ , then  $aM \leq \frac{b}{3} \cdot \frac{3q}{b} = q$  and we are done. Therefore, in what follows we can assume that  $a > \frac{b}{3}$ . We first consider the case when  $9 < \frac{3q}{c} < 10 < \frac{3q}{b} < 11$ .

As before, we can prove  $9(a-1) < 2b$ , and then  $10a < \frac{10 \times 5}{9 \times 4} \times 2b = \frac{50}{18}b < 3b < q$  as desired. Next we consider the case when  $10 < \frac{3q}{c} < 11 < \frac{3q}{b} < 12$ .

As before, we have  $10(a-1) < 2b$ , and then  $10a < \frac{11 \times 5}{10 \times 4} \times 2b = \frac{11}{4}b < 3b < q$  as desired. Finally, we deal with the case that

$$9 < \frac{3q}{c} < 10 < 11 < \frac{3q}{b} < 12. \quad (5.8)$$

Then we have  $6(a-1) < 2b$  and  $q > 11b/3$ . If  $q > 49$ , then  $b > q/4 > 343/4 > 85$ , and thus  $a > b/3 > 21$ . It follows that  $10a < \frac{10 \times 22}{6 \times 21} \times 6(a-1) < \frac{220}{63}b < \frac{11}{3}b < q$  as desired. If  $q = 49$ , in view of (5.8) we have  $12 < \frac{147}{12} < b < \frac{147}{11} < 14$  and  $14 < \frac{147}{10} < b < \frac{147}{9} < 16$ . Hence  $b = 13$ ,  $c = 15$  and thus  $a = c - b + 1 = 3$ , contradicting the assumption  $a > 4$ .

(vi) If  $l = 1$  and  $m_1 r = 5$ , then  $q = 5^k$ ,  $k \geq 2$  and

$$4 < \frac{2q}{c} < 5 < \frac{2q}{b} < 6.$$

Proposition 2.1 follows from Lemma 4.4.

This completes the proof.  $\square$

We remark that some part of the above proof works for the general case when  $(n, 6) = 1$ , and we are not aware of any counterexample to the general case. However, it seems to us that the method developed in this paper cannot be directly applied to solve the general case.

## 6. Connection with Dedekind sums

In this section, we deal with a related problem regarding Dedekind sums and prove the following result.

**Proposition 6.1.** *Let  $G$  be a finite cyclic group of order  $n$  with  $\gcd(n, 6) = 1$ , and let  $S = (x_1 g)(x_2 g)((n - x_3)g)((n - x_4)g)$  be a minimal zero-sum sequence, where  $g \in G$  with  $\text{ord}(g) = n$ ,  $1 \leq x_i < n$ ,  $i = 1, 2, 3, 4$  and  $x_1 + x_2 = x_3 + x_4$ . Then  $\text{ind}(S) = 1$  if and only if there exists an integer  $m$  such that  $\gcd(n, m) = 1$ ,  $1 \leq m < n$  and*

$$\left\lfloor \frac{mx_1}{n} \right\rfloor + \left\lfloor \frac{mx_2}{n} \right\rfloor \neq \left\lfloor \frac{mx_3}{n} \right\rfloor + \left\lfloor \frac{mx_4}{n} \right\rfloor. \quad (6.1)$$

**Proof.** Let  $a_i = \lfloor \frac{mx_i}{n} \rfloor$ ,  $i = 1, 2, 3, 4$ . Since  $|mx_i|_n = mx_i - a_i n$ ,  $i = 1, 2$  and  $|m(n - x_j)|_n = (a_j + 1) - a_j n$ ,  $j = 3, 4$ , we have

$$\begin{aligned} & |mx_1|_n + |mx_2|_n + |mx_3|_n + |mx_4|_n \\ &= mx_1 - a_1 n + mx_2 - a_2 n + (a_3 + 1)n - mx_3 + (a_4 + 1)n - mx_4 = (a_3 + a_4 - a_1 - a_2 + 2)n. \end{aligned}$$

From this we conclude that  $|mx_1|_n + |mx_2|_n + |mx_3|_n + |mx_4|_n = n$  if and only if  $\lfloor \frac{mx_1}{n} \rfloor + \lfloor \frac{mx_2}{n} \rfloor = \lfloor \frac{mx_3}{n} \rfloor + \lfloor \frac{mx_4}{n} \rfloor + 1$ .

Note that if  $x_1 + x_2 = x_3 + x_4$  and  $m$  is an integer such that  $\gcd(n, m) = 1$ ,  $1 \leq m < n$  and

$$\left\lfloor \frac{mx_3}{n} \right\rfloor + \left\lfloor \frac{mx_4}{n} \right\rfloor = \left\lfloor \frac{mx_1}{n} \right\rfloor + \left\lfloor \frac{mx_2}{n} \right\rfloor + 1,$$

then  $\gcd(n, n - m) = 1$  and

$$\left\lfloor \frac{(n - m)x_1}{n} \right\rfloor + \left\lfloor \frac{(n - m)x_2}{n} \right\rfloor = \left\lfloor \frac{(n - m)x_3}{n} \right\rfloor + \left\lfloor \frac{(n - m)x_4}{n} \right\rfloor + 1.$$

On the other hand,  $\lfloor \frac{mx_1}{n} \rfloor + \lfloor \frac{mx_2}{n} \rfloor \neq \lfloor \frac{mx_3}{n} \rfloor + \lfloor \frac{mx_4}{n} \rfloor$ ,  $1 \leq m < n$  and  $x_1 + x_2 = x_3 + x_4$  imply that

$$\left\lfloor \frac{mx_1}{n} \right\rfloor + \left\lfloor \frac{mx_2}{n} \right\rfloor = \left\lfloor \frac{mx_3}{n} \right\rfloor + \left\lfloor \frac{mx_4}{n} \right\rfloor \pm 1.$$

This completes the proof.  $\square$

Let  $h$  and  $k$  be positive integers with  $\gcd(h, k) = 1$ . Dedekind sums are defined by the equation

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k} \left( \frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right).$$

Let  $n$  be a positive integer such that for any integers  $x, m$ ,  $1 \leq x, m \leq n-1$  and  $\gcd(n, m) = 1$ . It is easy to prove that  $\left\lfloor \frac{mx}{n} \right\rfloor + \left\lfloor \frac{(n-m)x}{n} \right\rfloor = x-1$ . It follows that

$$\sum_{(n,m)=1, m=1}^{n-1} \left( \left\lfloor \frac{mx_1}{n} \right\rfloor + \left\lfloor \frac{mx_2}{n} \right\rfloor \right) = \sum_{(n,m)=1, m=1}^{n-1} \left( \left\lfloor \frac{mx_3}{n} \right\rfloor + \left\lfloor \frac{mx_4}{n} \right\rfloor \right)$$

when  $x_1 + x_2 = x_3 + x_4$ . Therefore, to prove (6.1), we need only evaluate the following sum

$$\sum_{(n,m)=1, m=1}^{n-1} \left( \left\lfloor \frac{mx_1}{n} \right\rfloor + \left\lfloor \frac{mx_2}{n} \right\rfloor - \left\lfloor \frac{mx_3}{n} \right\rfloor - \left\lfloor \frac{mx_4}{n} \right\rfloor \right)^2.$$

This sum can be computed by using Dedekind sums. However, in general, there is no simple formula for evaluating  $s(h, k)$  in detailed form.

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### Supplementary material

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